LECTURE V

THE THEORY OF CONTINUITY

The theory of continuity, with which we shall be occupied in the present lecture, is, in most of its refinements and developments, a purely mathematical subject--very beautiful, very important, and very delightful, but not, strictly speaking, a part of philosophy. The logical basis of the theory alone belongs to philosophy, and alone will occupy us to-night. The way the problem of continuity enters into philosophy is, broadly speaking, the following: Space and time are treated by mathematicians as consisting of points and instants, but they also have a property, easier to feel than to define, which is called continuity, and is thought by many philosophers to be destroyed when they are resolved into points and instants. Zeno, as we shall see, proved that analysis into points and instants was impossible if we adhered to the view that the number of points or instants in a finite space or time must be finite. Later philosophers, believing infinite number to be self-contradictory, have found here an antinomy: Spaces and times could not consist of a finite number of points and instants, for such reasons as Zeno's; they could not consist of an infinite number of points and instants, because infinite numbers were supposed to be self-contradictory. Therefore spaces and times, if real at all, must not be regarded as composed of points and instants.

But even when points and instants, as independent entities, are discarded, as they were by the theory advocated in our last lecture, the problems of continuity, as I shall try to show presently, remain, in a practically unchanged form. Let us therefore, to begin with, admit points and instants, and consider the problems in connection with this simpler or at least more familiar hypothesis.

The argument against continuity, in so far as it rests upon the supposed difficulties of infinite numbers, has been disposed of by the positive theory of the infinite, which will be considered in Lecture VII. But there remains a feeling--of the kind that led Zeno to the contention that the arrow in its flight is at rest--which suggests that points and instants, even if they are infinitely numerous, can only give a jerky motion, a succession of different immobilities, not the smooth transitions with which the senses have made us familiar. This feeling is due, I believe, to a failure to realise imaginatively, as well as abstractly, the nature of continuous series as they appear in mathematics. When a theory has been apprehended logically, there is often a long and serious labour still required in order to feel it: it is necessary to dwell upon it, to thrust out from the mind, one by one, the misleading suggestions of false but more familiar theories, to acquire the kind of intimacy which, in the case of a foreign language, would enable us to think and dream in it, not merely to construct laborious sentences by the help of grammar and dictionary. It is, I believe, the absence of this kind of intimacy which makes many philosophers regard the mathematical doctrine of continuity as an inadequate explanation of the continuity which we experience in the world of sense.

In the present lecture, I shall first try to explain in outline what the mathematical theory of continuity is in its philosophically important essentials. The application to actual space and time will not be in question to begin with. I do not see any reason to suppose that the points and instants which mathematicians introduce in dealing with space and time are actual physically existing entities, but I do see reason to suppose that the continuity of actual space and time may be more or less analogous to mathematical continuity. The theory of mathematical continuity is an abstract logical theory, not dependent for its validity upon any properties of actual space and time. What is claimed for it is that, when it is understood, certain characteristics of space and time, previously very hard to analyse, are found not to present any logical difficulty. What we know empirically about space and time is insufficient to enable us to decide between various mathematically possible alternatives, but these alternatives are all fully intelligible and fully adequate to the observed facts. For the present, however, it will be well to forget space and time and the continuity of sensible change, in order to return to these topics equipped with the weapons provided by the abstract theory of continuity.

Continuity, in mathematics, is a property only possible to a series of terms, i.e. to terms arranged in an order, so that we can say of any two that one comes before the other. Numbers in order of magnitude, the points on a line from left to right, the moments of time from earlier to later, are instances of series. The notion of order, which is here introduced, is one which is not required in the theory of cardinal number. It is possible to know that two classes have the same number of terms without knowing any order in which they are to be taken. We have an instance of this in such a case as English husbands and English wives: we can see that there must be the same number of husbands as of wives, without having to arrange them in a series. But continuity, which we are now to consider, is essentially a property of an order: it does not belong to a set of terms in themselves, but only to a set in a certain order. A set of terms which can be arranged in one order can always also be arranged in other orders, and a set of terms which can be arranged in a continuous order can always also be arranged in orders which are not continuous. Thus the essence of continuity must not be sought in the nature of the set of terms, but in the nature of their arrangement in a series.

Mathematicians have distinguished different degrees of continuity, and have confined the word "continuous," for technical purposes, to series having a certain high degree of continuity. But for philosophical purposes, all that is important in continuity is introduced by the lowest degree of continuity, which is called "compactness." A series is called "compact" when no two terms are consecutive, but between any two there are others. One of the simplest examples of a compact series is the series of fractions in order of magnitude. Given any two fractions, however near together, there are other fractions greater than the one and smaller than the other, and therefore no two fractions are consecutive. There is no fraction, for example, which is next after 1/2: if we choose some fraction which is very little greater than 1/2, say 51/100 we can find others, such as 101/200, which are nearer to 1/2. Thus between any two fractions, however little they differ, there are an infinite number of other fractions. Mathematical space and time also have this property of compactness, though whether actual space and time have it is a further question, dependent upon empirical evidence, and probably incapable of being answered with certainty.

In the case of abstract objects such as fractions, it is perhaps not very difficult to realise the logical possibility of their forming a compact series. The difficulties that might be felt are those of infinity, for in a compact series the number of terms between any two given terms must be infinite. But when these difficulties have been solved, the mere compactness in itself offers no great obstacle to the imagination. In more concrete cases, however, such as motion, compactness becomes much more repugnant to our habits of thought. It will therefore be desirable to consider explicitly the mathematical account of motion, with a view to making its logical possibility felt.
The mathematical account of motion is perhaps artificially simplified when regarded as describing what actually occurs in the physical world; but what actually occurs must be capable, by a certain amount of logical manipulation, of being brought within the scope of the mathematical account, and must, in its analysis, raise just such problems as are raised in their simplest form by this account. Neglecting, therefore,

for the present, the question of its physical adequacy, let us devote ourselves merely to considering its possibility as a formal statement of the nature of motion.

In order to simplify our problem as much as possible, let us imagine a tiny speck of light moving along a scale. What do we mean by saying that the motion is continuous? It is not necessary for our purposes to consider the whole of what the mathematician means by this statement: only part of what he means is philosophically important. One part of what he means is that, if we consider any two positions of the speck occupied at any two instants, there will be other intermediate positions occupied at intermediate instants. However near together we take the two positions, the speck will not jump suddenly from the one to the other, but will pass through an infinite number of other positions on the way. Every distance, however small, is traversed by passing through all the infinite series of positions between the two ends of the distance.

But at this point imagination suggests that we may describe the continuity of motion by saying that the speck always passes from one position at one instant to the next position at the next instant. As soon as we say this or imagine it, we fall into error, because there is no next point or next instant. If there were, we should find Zeno's paradoxes, in some form, unavoidable, as will appear in our next lecture. One simple paradox may serve as an illustration. If our speck is in motion along the scale throughout the whole of a certain time, it cannot be at the same point at two consecutive instants. But it cannot, from one instant to the next, travel further than from one point to the next, for if it did, there would be no instant at which it was in the positions intermediate between that at the first instant and that at the next, and we agreed that the continuity of motion excludes the possibility of such sudden jumps. It follows that our speck must, so long as it moves, pass from one point at one instant to the next point at the next instant. Thus there will be just one perfectly definite velocity with which all motions must take place: no motion can be faster than this, and no motion can be slower. Since this conclusion is false, we must reject the hypothesis upon which it is based, namely that there are consecutive points and instants.[18] Hence the continuity of motion must not be supposed to consist in a body's occupying consecutive positions at consecutive times.

[18] The above paradox is essentially the same as Zeno's argument of the stadium which will be considered in our next lecture.

The difficulty to imagination lies chiefly, I think, in keeping out the suggestion of infinitesimal distances and times. Suppose we halve a given distance, and then halve the half, and so on, we can continue the process as long as we please, and the longer we continue it, the smaller the resulting distance becomes. This infinite divisibility seems, at first sight, to imply that there are infinitesimal distances, i.e. distances so small that any finite fraction of an inch would be greater. This, however, is an error. The continued bisection of our distance, though it gives us continually smaller distances, gives us always finite distances. If our original distance was an inch, we reach successively half an inch, a quarter of an inch, an eighth, a sixteenth, and so on; but every one of this infinite series of diminishing distances is finite. "But," it may be said, "in the end the distance will grow infinitesimal." No, because there is no end. The process of bisection is one which can, theoretically, be carried on for ever, without any last term being attained. Thus infinite divisibility of distances, which must be admitted, does not imply that there are distances so small that any finite distance would be larger.

It is easy, in this kind of question, to fall into an elementary logical blunder. Given any finite distance, we can find a smaller distance; this may be expressed in the ambiguous form "there is a distance smaller than any finite distance." But if this is then interpreted as meaning "there is a distance such that, whatever finite distance may be chosen, the distance in question is smaller," then the statement is false. Common language is ill adapted to expressing matters of this kind, and philosophers who have been dependent on it have frequently been misled by it.

In a continuous motion, then, we shall say that at any given instant the moving body occupies a certain position, and at other instants it occupies other positions; the interval between any two instants and between any two positions is always finite, but the continuity of the motion is shown in the fact that, however near together we take the two positions and the two instants, there are an infinite number of positions still nearer together, which are occupied at instants that are also still nearer together. The moving body never jumps from one position to another, but always passes by a gradual transition through an infinite number of intermediaries. At a given instant, it is where it is, like Zeno's arrow;[19] but we cannot say that it is at rest at the instant, since the instant does not last for a finite time, and there is not a beginning and end of the instant with an interval between them. Rest consists in being in the same position at all the instants throughout a certain finite period, however short; it does not consist simply in a body's being where it is at a given instant. This whole theory, as is obvious, depends upon the nature of compact series, and demands, for its full comprehension, that compact series should have become familiar and easy to the imagination as well as to deliberate thought.

[19] See next lecture.

What is required may be expressed in mathematical language by saying that the position of a moving body must be a continuous function of the time. To define accurately what this means, we proceed as follows. Consider a particle which, at the moment t, is at the point P. Choose now any small portion P1P2 of the path of the particle, this portion being one which contains P. We say then that, if the motion of the particle is continuous at the time t, it must be possible to find two instants t1, t2, one earlier than t and one later, such that throughout the whole time from t1 to t2 (both included), the particle lies between P1 and P2. And we say that this must still hold however small we make the portion P1P2. When this is the case, we say that the motion is continuous at the time t; and when the motion is continuous at all times, we say that the motion as a whole is continuous. It is obvious that if the particle were to jump suddenly from P to some other point Q, our definition would fail for all intervals P1P2 which were too small to include Q. Thus our definition affords an analysis of the continuity of motion, while admitting points and instants and denying infinitesimal distances in space or periods in time.

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      P1    P     P2    Q
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Philosophers, mostly in ignorance of the mathematician's analysis, have adopted other and more heroic methods of dealing with the primâ facie difficulties of continuous motion. A typical and recent example of philosophic theories of motion is afforded by Bergson, whose views on this subject I have examined elsewhere.[20]

[20] Monist, July 1912, pp. 337-341.

Apart from definite arguments, there are certain feelings, rather than reasons, which stand in the way of an acceptance of the mathematical account of motion. To begin with, if a body is moving at all fast, we see its motion just as we see its colour. A slow motion, like that of the hour-hand of a watch, is only known in the way which mathematics would lead us to expect, namely by observing a change of position after a lapse of time; but, when we observe the motion of the second-hand, we do not merely see first one position and then another--we see something as directly sensible as colour. What is this something that we see, and that we call visible motion? Whatever it is, it is not the successive occupation of successive positions: something beyond the mathematical theory of motion is required to account for it. Opponents of the mathematical theory emphasise this fact. "Your theory," they say, "may be very logical, and might apply admirably to some other world; but in this actual world, actual motions are quite different from what your theory would declare them to be, and require, therefore, some different philosophy from yours for their adequate explanation."

The objection thus raised is one which I have no wish to underrate, but I believe it can be fully answered without departing from the methods and the outlook which have led to the mathematical theory of motion. Let us, however, first try to state the objection more fully.

If the mathematical theory is adequate, nothing happens when a body moves except that it is in different places at different times. But in this sense the hour-hand and the second-hand are equally in motion, yet in the second-hand there is something perceptible to our senses which is absent in the hour-hand. We can see, at each moment, that the second-hand is moving, which is different from seeing it first in one place and then in another. This seems to involve our seeing it simultaneously in a number of places, although it must also involve our seeing that it is in some of these places earlier than in others. If, for example, I move my hand quickly from left to right, you seem to see the whole movement at once, in spite of the fact that you know it begins at the left and ends at the right. It is this kind of consideration, I think, which leads Bergson and many others to regard a movement as really one indivisible whole, not the series of separate states imagined by the mathematician.

To this objection there are three supplementary answers, physiological, psychological, and logical. We will consider them successively.

(1) The physiological answer merely shows that, if the physical world is what the mathematician supposes, its sensible appearance may nevertheless be expected to be what it is. The aim of this answer is thus the modest one of showing that the mathematical account is not impossible as applied to the physical world; it does not even attempt to show that this account is necessary, or that an analogous account applies in psychology.

When any nerve is stimulated, so as to cause a sensation, the sensation does not cease instantaneously with the cessation of the stimulus, but dies away in a short finite time. A flash of lightning, brief as it is to our sight, is briefer still as a physical phenomenon: we continue to see it for a few moments after the light-waves have ceased to strike the eye. Thus in the case of a physical motion, if it is sufficiently swift, we shall actually at one instant see the moving body throughout a finite portion of its course, and not only at the exact spot where it is at that instant. Sensations, however, as they die away, grow gradually fainter; thus the sensation due to a stimulus which is recently past is not exactly like the sensation due to a present stimulus. It follows from this that, when we see a rapid motion, we shall not only see a number of positions of the moving body simultaneously, but we shall see them with different degrees of intensity--the present position most vividly, and the others with diminishing vividness, until sensation fades away into immediate memory. This state of things accounts fully for the perception of motion. A motion is perceived, not merely inferred, when it is sufficiently swift for many positions to be sensible at one time; and the earlier and later parts of one perceived motion are distinguished by the less and greater vividness of the sensations.

This answer shows that physiology can account for our perception of motion. But physiology, in speaking of stimulus and sense-organs and a physical motion distinct from the immediate object of sense, is assuming the truth of physics, and is thus only capable of showing the physical account to be possible, not of showing it to be necessary. This consideration brings us to the psychological answer.

(2) The psychological answer to our difficulty about motion is part of a vast theory, not yet worked out, and only capable, at present, of being vaguely outlined. We considered this theory in the third and fourth

lectures; for the present, a mere sketch of its application to our present problem must suffice. The world of physics, which was assumed in the physiological answer, is obviously inferred from what is given in sensation; yet as soon as we seriously consider what is actually given in sensation, we find it apparently very different from the world of physics. The question is thus forced upon us: Is the inference from sense to physics a valid one? I believe the answer to be affirmative, for reasons which I suggested in the third and fourth lectures; but the answer cannot be either short or easy. It consists, broadly speaking, in showing that, although the particles, points, and instants with which physics operates are not themselves given in experience, and are very likely not actually existing things, yet, out of the materials provided in sensation, it is possible to make logical constructions having the mathematical properties which physics assigns to particles, points, and instants. If this can be done, then all the propositions of physics can be translated, by a sort of dictionary, into propositions about the kinds of objects which are given in sensation.

Applying these general considerations to the case of motion, we find that, even within the sphere of immediate sense-data, it is necessary, or at any rate more consonant with the facts than any other equally simple view, to distinguish instantaneous states of objects, and to regard such states as forming a compact series. Let us consider a body which is moving swiftly enough for its motion to be perceptible, and long enough for its motion to be not wholly comprised in one sensation. Then, in spite of the fact that we see a finite extent of the motion at one instant, the extent which we see at one instant is different from that which we see at another. Thus we are brought back, after all, to a series of momentary views of the moving body, and this series will be compact, like the former physical series of points. In fact, though the terms of the series seem different, the mathematical character of the series is unchanged, and the whole mathematical theory of motion will apply to it verbatim.

When we are considering the actual data of sensation in this connection, it is important to realise that two sense-data may be, and must sometimes be, really different when we cannot perceive any difference between them. An old but conclusive reason for believing this was emphasised by Poincaré.[21] In all cases of sense-data capable of gradual change, we may find one sense-datum indistinguishable from another, and that other indistinguishable from a third, while yet the first and third are quite easily distinguishable. Suppose, for example, a person with his eyes shut is holding a weight in his hand, and someone noiselessly adds a small extra weight. If the extra weight is small enough, no difference will be perceived in the sensation. After a time, another small extra weight may be added, and still no change will be perceived; but if both extra weights had been added at once, it may be that the change would be quite easily perceptible. Or, again, take shades of colour. It would be easy to find three stuffs of such closely similar shades that no difference could be perceived between the first and second, nor yet between the second and third, while yet the first and third would be distinguishable. In such a case, the second shade cannot be the same as the first, or it would be distinguishable from the third; nor the same as the third, or it would be distinguishable from the first. It must, therefore, though indistinguishable from both, be really intermediate between them.

[21] "Le continu mathématique," Revue de Métaphysique et de Morale, vol. i. p. 29.

Such considerations as the above show that, although we cannot distinguish sense-data unless they differ by more than a certain amount, it is perfectly reasonable to suppose that sense-data of a given kind, such as weights or colours, really form a compact series. The objections which may be brought from a psychological point of view against the mathematical theory of motion are not, therefore, objections to this theory properly understood, but only to a quite unnecessary assumption of simplicity in the momentary object of sense. Of the immediate object of sense, in the case of a visible motion, we may say that at each instant it is in all the positions which remain sensible at that instant; but this set of positions changes continuously from moment to moment, and is amenable to exactly the same mathematical treatment as if it were a mere point. When we assert that some mathematical account of phenomena is correct, all that we primarily assert is that something definable in terms of the crude phenomena satisfies our formulæ; and in this sense the mathematical theory of motion is applicable to the data of sensation as well as to the supposed particles of abstract physics.

There are a number of distinct questions which are apt to be confused when the mathematical continuum is said to be inadequate to the facts of sense. We may state these, in order of diminishing generality, as follows:--

(a) Are series possessing mathematical continuity logically possible?

(b) Assuming that they are possible logically, are they not impossible as applied to actual sense-data, because, among actual sense-data, there are no such fixed mutually external terms as are to be found, e.g., in the series of fractions?

(c) Does not the assumption of points and instants make the whole mathematical account fictitious?

(d) Finally, assuming that all these objections have been answered, is there, in actual empirical fact, any sufficient reason to believe the world of sense continuous?

Let us consider these questions in succession.

(a) The question of the logical possibility of the mathematical continuum turns partly on the elementary misunderstandings we considered at the beginning of the present lecture, partly on the possibility of the mathematical infinite, which will occupy our next two lectures, and partly on the logical form of the answer to the Bergsonian objection which we stated a few minutes ago. I shall say no more on this topic at present, since it is desirable first to complete the psychological answer.

(b) The question whether sense-data are composed of mutually external units is not one which can be decided by empirical evidence. It is often urged that, as a matter of immediate experience, the sensible flux is devoid of divisions, and is falsified by the dissections of the intellect. Now I have no wish to argue that this view is contrary to immediate experience: I wish only to maintain that it is essentially incapable of being proved by immediate experience. As we saw, there must be among sense-data differences so slight as to be imperceptible: the fact that sense-data are immediately given does not mean that their differences also must be immediately given (though they may be). Suppose, for example, a coloured surface on which the colour changes gradually--so gradually that the difference of colour in two very neighbouring portions is imperceptible, while the difference between more widely separated portions is quite noticeable. The effect produced, in such a case, will be precisely that of "interpenetration," of transition which is not a matter of discrete units. And since it tends to be supposed that the colours, being immediate data, must appear different if they are different, it seems easily to follow that "interpenetration" must be the ultimately right account. But this does not follow. It is unconsciously assumed, as a premiss for a reductio ad absurdum of the analytic view, that, if A and B are immediate data, and

A differs from B, then the fact that they differ must also be an immediate datum. It is difficult to say how this assumption arose, but I think it is to be connected with the confusion between "acquaintance" and "knowledge about." Acquaintance, which is what we derive from sense, does not, theoretically at least, imply even the smallest "knowledge about," i.e. it does not imply knowledge of any proposition concerning the object with which we are acquainted. It is a mistake to speak as if acquaintance had degrees: there is merely acquaintance and non-acquaintance. When we speak of becoming "better acquainted," as for instance with a person, what we must mean is, becoming acquainted with more parts of a certain whole; but the acquaintance with each part is either complete or nonexistent. Thus it is a mistake to say that if we were perfectly acquainted with an object we should know all about it. "Knowledge about" is knowledge of propositions, which is not involved necessarily in acquaintance with the constituents of the propositions. To know that two shades of colour are different is knowledge about them; hence acquaintance with the two shades does not in any way necessitate the knowledge that they are different.

From what has just been said it follows that the nature of sense-data cannot be validly used to prove that they are not composed of mutually external units. It may be admitted, on the other hand, that nothing in their empirical character specially necessitates the view that they are composed of mutually external units. This view, if it is held, must be held on logical, not on empirical, grounds. I believe that the logical grounds are adequate to the conclusion. They rest, at bottom, upon the impossibility of explaining complexity without assuming constituents. It is undeniable that the visual field, for example, is complex; and so far as I can see, there is always self-contradiction in the theories which, while admitting this complexity, attempt to deny that it results from a combination of mutually external units. But to pursue this topic would lead us too far from our theme, and I shall therefore say no more about it at present.

(c) It is sometimes urged that the mathematical account of motion is rendered fictitious by its assumption of points and instants. Now there are here two different questions to be distinguished. There is the question of absolute or relative space and time, and there is the question whether what occupies space and time must be composed of elements which have no extension or duration. And each of these questions in turn may take two forms, namely: (α) is the hypothesis consistent with the facts and with logic? (β) is it necessitated by the facts or by logic? I wish to answer, in each case, yes to the first form of the question, and no to the second. But in any case the mathematical account of motion will not be fictitious, provided a right interpretation is given to the words "point" and "instant." A few words on each alternative will serve to make this clear.

Formally, mathematics adopts an absolute theory of space and time, i.e. it assumes that, besides the things which are in space and time, there are also entities, called "points" and "instants," which are occupied by things. This view, however, though advocated by Newton, has long been regarded by mathematicians as merely a convenient fiction. There is, so far as I can see, no conceivable evidence either for or against it. It is logically possible, and it is consistent with the facts. But the facts are also consistent with the denial of spatial and temporal entities over and above things with spatial and temporal relations. Hence, in accordance with Occam's razor, we shall do well to abstain from either assuming or denying points and instants. This means, so far as practical working out is concerned, that we adopt the relational theory; for in practice the refusal to assume points and instants has the same effect as the denial of them. But in strict theory the two are quite different, since the denial introduces an element of unverifiable dogma which is wholly absent when we merely refrain from the assertion. Thus, although we shall derive points and instants from things, we shall leave the bare possibility open that they may also have an independent existence as simple entities.

We come now to the question whether the things in space and time are to be conceived as composed of elements without extension or duration, i.e. of elements which only occupy a point and an instant. Physics, formally, assumes in its differential equations that things consist of elements which occupy only a point at each instant, but persist throughout time. For reasons explained in Lecture IV., the persistence of things through time is to be regarded as the formal result of a logical construction, not as necessarily implying any actual persistence. The same motives, in fact, which lead to the division of things into point-particles, ought presumably to lead to their division into instant-particles, so that the ultimate formal constituent of the matter in physics will be a point-instant-particle. But such objects, as well as the particles of physics, are not data. The same economy of hypothesis, which dictates the practical adoption of a relative rather than an absolute space and time, also dictates the practical adoption of material elements which have a finite extension and duration. Since, as we saw in Lecture IV., points and instants can be constructed as logical functions of such elements, the mathematical account of motion, in which a particle passes continuously through a continuous series of points, can be interpreted in a form which assumes only elements which agree with our actual data in having a finite extension and duration. Thus, so far as the use of points and instants is concerned, the mathematical account of motion can be freed from the charge of employing fictions.

(d) But we must now face the question: Is there, in actual empirical fact, any sufficient reason to believe the world of sense continuous? The answer here must, I think, be in the negative. We may say that the hypothesis of continuity is perfectly consistent with the facts and with logic, and that it is technically simpler than any other tenable hypothesis. But since our powers of discrimination among very similar sensible objects are not infinitely precise, it is quite impossible to decide between different theories which only differ in regard to what is below the margin of discrimination. If, for example, a coloured surface which we see consists of a finite number of very small surfaces, and if a motion which we see consists, like a cinematograph, of a large finite number of successive positions, there will be nothing empirically discoverable to show that objects of sense are not continuous. In what is called experienced continuity, such as is said to be given in sense, there is a large negative element: absence of perception of difference occurs in cases which are thought to give perception of absence of difference. When, for example, we cannot distinguish a colour A from a colour B, nor a colour B from a colour C, but can distinguish A from C, the indistinguishability is a purely negative fact, namely, that we do not perceive a difference. Even in regard to immediate data, this is no reason for denying that there is a difference. Thus, if we see a coloured surface whose colour changes gradually, its sensible appearance if the change is continuous will be indistinguishable from what it would be if the change were by small finite jumps. If this is true, as it seems to be, it follows that there can never be any empirical evidence to demonstrate that the sensible world is continuous, and not a collection of a very large finite number of elements of which each differs from its neighbour in a finite though very small degree. The continuity of space and time, the infinite number of different shades in the spectrum, and so on, are all in the nature of unverifiable hypotheses--perfectly possible logically, perfectly consistent with the known facts, and simpler technically than any other tenable hypotheses, but not the sole hypotheses which are logically and empirically adequate.

If a relational theory of instants is constructed, in which an "instant" is defined as a group of events simultaneous with each other and not all simultaneous with any event outside the group, then if our resulting series of instants is to be compact, it must be possible, if x wholly precedes y, to find an event z, simultaneous with part of x, which wholly precedes some event which wholly precedes y. Now this requires that the number of events concerned should be infinite in any finite period of time. If this is to be the case in the world of one man's sense-data, and if each sense-datum is to have not less than a certain finite temporal extension, it will be necessary to assume that we always have an infinite number of sense-data simultaneous with any given sense-datum. Applying similar considerations to space, and assuming that sense-data are to have not less than a certain spatial extension, it will be necessary to suppose that an infinite number of sense-data overlap spatially with any given sense-datum. This hypothesis is possible, if we suppose a single sense-datum, e.g. in sight, to be a finite surface, enclosing other surfaces which are also single sense-data. But there are difficulties in such a hypothesis, and I do not know whether these difficulties could be successfully met. If they cannot, we must do one of two things: either declare that the world of one man's sense-data is not continuous, or else refuse to admit that there is any lower limit to the duration and extension of a single sense-datum. I do not know what is the right course to adopt as regards these alternatives. The logical analysis we have been considering provides the apparatus for dealing with the various hypotheses, and the empirical decision between them is a problem for the psychologist.

(3) We have now to consider the logical answer to the alleged difficulties of the mathematical theory of motion, or rather to the

positive theory which is urged on the other side. The view urged explicitly by Bergson, and implied in the doctrines of many philosophers, is, that a motion is something indivisible, not validly analysable into a series of states. This is part of a much more general doctrine, which holds that analysis always falsifies, because the parts of a complex whole are different, as combined in that whole, from what they would otherwise be. It is very difficult to state this doctrine in any form which has a precise meaning. Often arguments are used which have no bearing whatever upon the question. It is urged, for example, that when a man becomes a father, his nature is altered by the new relation in which he finds himself, so that he is not strictly identical with the man who was previously not a father. This may be true, but it is a causal psychological fact, not a logical fact. The doctrine would require that a man who is a father cannot be strictly identical with a man who is a son, because he is modified in one way by the relation of fatherhood and in another by that of sonship. In fact, we may give a precise statement of the doctrine we are combating in the form: There can never be two facts concerning the same thing. A fact concerning a thing always is or involves a relation to one or more entities; thus two facts concerning the same thing would involve two relations of the same thing. But the doctrine in question holds that a thing is so modified by its relations that it cannot be the same in one relation as in another. Hence, if this doctrine is true, there can never be more than one fact concerning any one thing. I do not think the philosophers in question have realised that this is the precise statement of the view they advocate, because in this form the view is so contrary to plain truth that its falsehood is evident as soon as it is stated. The discussion of this question, however, involves so many logical subtleties, and is so beset with difficulties, that I shall not pursue it further at present.

When once the above general doctrine is rejected, it is obvious that, where there is change, there must be a succession of states. There cannot be change--and motion is only a particular case of change--unless there is something different at one time from what there is at some other time. Change, therefore, must involve relations and complexity, and must demand analysis. So long as our analysis has only gone as far as other smaller changes, it is not complete; if it is to be complete, it must end with terms that are not changes, but are related by a relation of earlier and later. In the case of changes which appear continuous, such as motions, it seems to be impossible to find anything other than change so long as we deal with finite periods of time, however short. We are thus driven back, by the logical necessities of the case, to the conception of instants without duration, or at any rate without any duration which even the most delicate instruments can reveal. This conception, though it can be made to seem difficult, is really easier than any other that the facts allow. It is a kind of logical framework into which any tenable theory must fit--not necessarily itself the statement of the crude facts, but a form in which statements which are true of the crude facts can be made by a suitable interpretation. The direct consideration of the crude facts of the physical world has been undertaken in earlier lectures; in the present lecture, we have only been concerned to show that nothing in the crude facts is inconsistent with the mathematical doctrine of continuity, or demands a continuity of a radically different kind from that of mathematical motion.