Chapter XXIV.

Of The Remaining Laws Of Nature.

§ 1. In the First Book we found that all the assertions which can be conveyed by language, express some one or more of five different things: Existence; Order in Place; Order in Time; Causation; and Resemblance.(197) Of these, Causation, in our view of the subject, not being fundamentally different from Order in Time, the five species of possible assertions are reduced to four. The propositions which affirm Order in Time in either of its two modes, Co-existence and Succession, have formed, thus far, the subject of the present Book. And we have now concluded the exposition, so far as it falls within the limits assigned to this work, of the nature of the evidence on which these propositions rest, and the processes of investigation by which they are ascertained and proved. There remain three classes of facts: Existence, Order in Place, and Resemblance; in regard to which the same questions are now to be resolved.

Regarding the first of these, very little needs be said. Existence in general, is a subject not for our science, but for metaphysics. To determine what things can be recognized as really existing, independently of our own sensible or other impressions, and in what meaning the term is, in that case, predicated of them, belongs to the consideration of "Things in themselves," from which, throughout this work, we have as much as possible kept aloof. Existence, so far as Logic is concerned about it, has reference only to phenomena; to actual, or possible, states of external or internal consciousness, in ourselves or others. Feelings of sensitive beings, or possibilities of having such feelings, are the only things the existence of which can be a subject of logical induction, because the only things of which the existence in individual cases can be a subject of experience.

It is true that a thing is said by us to exist, even when it is absent, and therefore is not and can not be perceived. But even then, its existence is to us only another word for our conviction that we should perceive it on a certain supposition; namely, if we were in the needful circumstances of time and place, and endowed with the needful perfection of organs. My belief that the Emperor of China exists, is simply my belief that if I were transported to the imperial palace or some other locality in Pekin, I should see him. My belief that Julius Cæsar existed, is my belief that I should have seen him if I had been present in the field of Pharsalia, or in the senate-house at Rome. When I believe that stars exist beyond the utmost range of my vision, though assisted by the most powerful telescopes yet invented, my belief, philosophically expressed, is, that with still better telescopes, if such existed, I could see them, or that they may be perceived by beings less remote from them in space, or whose capacities of perception are superior to mine.

The existence, therefore, of a phenomenon, is but another word for its being perceived, or for the inferred possibility of perceiving it. When the phenomenon is within the range of present observation, by present observation we assure ourselves of its existence; when it is beyond that range, and is therefore said to be absent, we infer its existence from marks or evidences. But what can these evidences be? Other phenomena; ascertained by induction to be connected with the given phenomenon, either in the way of succession or of co-existence. The simple existence, therefore, of an individual phenomenon, when not directly perceived, is inferred from some inductive law of succession or co-existence; and is consequently not amenable to any peculiar inductive principles. We prove the existence of a thing, by proving that it is connected by succession or co-existence with some known thing.

With respect to *general* propositions of this class, that is, which affirm the bare fact of existence, they have a peculiarity which renders the logical treatment of them a very easy matter; they are generalizations which are sufficiently proved by a single instance. That ghosts, or unicorns, or sea-serpents exist, would be fully established if it could be ascertained positively that such things had been even once seen. Whatever has once happened, is capable of happening again; the only question relates to the conditions under which it happens.

So far, therefore, as relates to simple existence, the Inductive Logic has no knots to untie. And we may proceed to the remaining two of the great classes into which facts have been divided; Resemblance, and Order in Place.

§ 2. Resemblance and its opposite, except in the case in which they assume the names of Equality and Inequality, are seldom regarded as subjects of science; they are supposed to be perceived by simple apprehension; by merely applying our senses or directing our attention to the two objects at once, or in immediate succession. And this simultaneous, or virtually simultaneous, application of our faculties to the two things which are to be compared, does necessarily constitute the ultimate appeal, wherever such application is practicable. But, in most cases, it is not practicable: the objects can not be brought so close together that the feeling of their resemblance (at least a complete feeling of it) directly arises in the mind. We can only compare each of them with some third object, capable of being transported from one to the other. And besides, even when the objects can be brought into immediate juxtaposition, their resemblance or difference is but imperfectly known to us, unless we have compared them minutely, part by part. Until this has been done, things in reality very dissimilar often appear undistinguishably alike. Two lines of very unequal length will appear about equal when lying in different directions; but place them parallel with their farther extremities even, and if we look at the nearer extremities, their inequality becomes a matter of direct perception.

To ascertain whether, and in what, two phenomena resemble or differ, is not always, therefore, so easy a thing as it might at first appear. When the two can not be brought into juxtaposition, or not so that the observer is able to compare their several parts in detail, he must employ the indirect means of reasoning and general propositions. When we can not bring two straight lines together, to determine whether they are equal, we do it by the physical aid of a foot-rule applied first to one and then to the other, and the logical aid of the general proposition or formula, "Things which are equal to the same thing are equal to one another." The comparison of two things through the intervention of a third thing, when their direct comparison is impossible, is the appropriate scientific process for ascertaining resemblances and dissimilarities, and is the sum total of what Logic has to teach on the subject.

An undue extension of this remark induced Locke to consider reasoning itself as nothing but the comparison of two ideas through the medium of a third, and knowledge as the perception of the agreement or disagreement of two ideas; doctrines which the Condillac school blindly adopted, without the qualifications and distinctions with which they were studiously guarded by their illustrious author. Where, indeed, the agreement or disagreement (otherwise called resemblance or dissimilarity) of any two things is the very matter to be determined, as is the case particularly in the sciences of quantity and extension; there, the process by which a solution, if not attainable by direct perception, must be indirectly sought, consists in comparing these two things through the medium of a third. But this is far from being true of all inquiries. The knowledge that bodies fall to the ground is not a perception of agreement or disagreement, but of a series of physical occurrences, a succession of sensations. Locke's definitions of knowledge and of reasoning required to be limited to our knowledge of, and reasoning about, resemblances. Nor, even when thus restricted, are the propositions strictly correct; since the comparison is not made, as he represents, between the ideas of the two phenomena, but between the phenomena themselves. This mistake has been pointed out in an earlier part of our inquiry,(198) and we traced it to an imperfect conception of what takes place in mathematics, where very often the comparison is really made between the ideas, without any appeal to the outward senses; only, however, because in mathematics a comparison of the ideas is strictly equivalent to a comparison of the phenomena themselves. Where, as in the case of numbers, lines, and figures, our idea of an object is a complete picture of the object, so far as respects the matter in hand; we can, of course, learn from the picture, whatever could be learned from the object itself by mere contemplation of it as it exists at the particular instant when the picture is taken. No mere contemplation of gunpowder would ever teach us that a spark would make it explode, nor, consequently, would the contemplation of the idea of gunpowder do so; but the mere contemplation of a straight line shows that it can not inclose a space; accordingly the contemplation of the idea of it will show the same. What takes place in mathematics is thus no argument that the comparison is between the ideas only. It is always, either indirectly or directly, a comparison of the phenomena.

In cases in which we can not bring the phenomena to the test of direct inspection at all, or not in a manner sufficiently precise, but must judge of their resemblance by inference from other resemblances or dissimilarities more accessible to observation, we of course require, as in all cases of ratiocination, generalizations or formulæ applicable to the subject. We must reason from laws of nature; from the uniformities which are observable in the fact of likeness or unlikeness.

§ 3. Of these laws or uniformities, the most comprehensive are those supplied by mathematics; the axioms relating to equality, inequality, and proportionality, and the various theorems thereon founded. And these are the only Laws of Resemblance which require to be, or which can be, treated apart. It is true there are innumerable other theorems which affirm resemblances among phenomena; as that the angle of the reflection of light is *equal* to its angle of incidence (equality being merely exact resemblance in magnitude). Again, that the heavenly bodies describe *equal* areas in equal times; and that their periods of revolution are *proportional* (another species of resemblance) to the sesquiplicate powers of their distances from the centre of force. These and similar propositions affirm resemblances, of the same nature with those asserted in the theorems of mathematics; but the distinction is, that the propositions of mathematics are true of all phenomena whatever, or at least without distinction of origin; while the truths in question are affirmed only of special phenomena, which originate in a certain way; and the equalities, proportionalities, or other resemblances, which exist between such phenomena, must necessarily be either derived from, or identical with, the law of their origin--the law of causation on which they depend. The equality of the areas described in equal times by the planets, is *derived* from the laws of the causes; and, until its derivation was shown, it was an empirical law. The equality of the angles of reflection and incidence is *identical* with the law of the cause; for the cause is the incidence of a ray of light upon a reflecting surface, and the equality in question is the very law according to which that cause produces its effects. This class, therefore, of the uniformities of resemblance between phenomena, are inseparable, in fact and in thought, from the laws of the production of those phenomena; and the principles of induction applicable to them are no other than those of which we have treated in the preceding chapters of this Book.

It is otherwise with the truths of mathematics. The laws of equality and inequality between spaces, or between numbers, have no connection with laws of causation. That the angle of reflection is equal to the angle of incidence, is a statement of the mode of action of a particular cause; but that when two straight lines intersect each other the opposite angles are equal, is true of all such lines and angles, by whatever cause produced. That the squares of the periodic times of the planets are proportional to the cubes of their distances from the sun, is a uniformity derived from the laws of the causes (or forces) which produce the planetary motions; but that the square of any number is four times the square of half the number, is true independently of any cause. The only laws of resemblance, therefore, which we are called upon to consider independently of causation, belong to the province of mathematics.

§ 4. The same thing is evident with respect to the only one remaining of our five categories, Order in Place. The order in place, of the effects of a cause, is (like every thing else belonging to the effects) a consequence of the laws of that cause. The order in place, or, as we have termed it, the collocation, of the primeval causes, is (as well as their resemblance) in each instance an ultimate fact, in which no laws or uniformities are traceable. The only remaining general propositions respecting order in place, and the only ones which have nothing to do with causation, are some of the truths of geometry; laws through which we are able, from the order in place of certain points, lines, or spaces, to infer the order in place of others which are connected with the former in some known mode; quite independently of the particular nature of those points, lines, or spaces, in any other respect than position or magnitude, as well as independently of the physical cause from which in any particular case they happen to derive their origin.

It thus appears that mathematics is the only department of science into the methods of which it still remains to inquire. And there is the less necessity that this inquiry should occupy us long, as we have already, in the Second Book, made considerable progress in it. We there remarked, that the directly inductive truths of mathematics are few in number; consisting of the axioms, together with certain propositions concerning existence, tacitly involved in most of the so-called definitions. And we gave what appeared conclusive reasons for affirming that these original premises, from which the remaining truths of the science are deduced, are, notwithstanding all appearances to the contrary, results of observation and experience; founded, in short, on the evidence of the senses. That things equal to the same thing are equal to one another, and that two straight lines which have once intersected one another continue to diverge, are inductive truths; resting, indeed, like the law of universal causation, only on induction *per enumerationem simplicem*; on the fact that they have been perpetually perceived to be true, and never once found to be false. But, as we have seen in a recent chapter that this evidence, in the case of a law so completely universal as the law of causation, amounts to the fullest proof, so is this even more evidently true of the general propositions to which we are now adverting; because, as a perception of their truth in any individual case whatever, requires only the simple act of looking at the objects in a proper position, there never could have been in their case (what, for a long period, there were in the case of the law of causation) instances which were apparently, though not really, exceptions to them. Their infallible truth was recognized from the very dawn of speculation; and as their extreme familiarity made it impossible for the mind to conceive the objects under any other law, they were, and still are, generally considered as truths recognized by their own evidence, or by instinct.

§ 5. There is something which seems to require explanation, in the fact that the immense multitude of truths (a multitude still as far from being exhausted as ever) comprised in the mathematical sciences, can be elicited from so small a number of elementary laws. One sees not, at first, how it is that there can be room for such an infinite variety of true propositions, on subjects apparently so limited.

To begin with the science of number. The elementary or ultimate truths of this science are the common axioms concerning equality, namely, "Things which are equal to the same thing are equal to one another," and "Equals added to equals make equal sums" (no other axioms are required),(199) together with the definitions of the various numbers. Like other so-called definitions, these are composed of two things, the explanation of a name, and the assertion of a fact; of which the latter alone can form a first principle or premise of a science. The fact asserted in the definition of a number is a physical fact. Each of the numbers two, three, four, etc., denotes physical phenomena, and connotes a physical property of those phenomena. Two, for instance, denotes all pairs of things, and twelve all dozens of things, connoting what makes them pairs, or dozens; and that which makes them so is something physical; since it can not be denied that two apples are physically distinguishable from three apples, two horses from one horse, and so forth; that they are a different visible and tangible phenomenon. I am not undertaking to say what the difference is; it is enough that there is a difference of which the senses can take cognizance. And although a hundred and two horses are not so easily distinguished from a hundred and three, as two horses are from three--though in most positions the senses do not perceive any difference--yet they may be so placed that a difference will be perceptible, or else we should never have distinguished them, and given them different names. Weight is confessedly a physical property of things; yet small differences between great weights are as imperceptible to the senses in most situations, as small differences between great numbers; and are only put in evidence by placing the two objects in a peculiar position--namely, in the opposite scales of a delicate balance.

What, then, is that which is connoted by a name of number? Of course, some property belonging to the agglomeration of things which we call by the name; and that property is, the characteristic manner in which the agglomeration is made up of, and may be separated into, parts. I will endeavor to make this more intelligible by a few explanations.

When we call a collection of objects *two*, *three*, or *four*, they are not two, three, or four in the abstract; they are two, three, or four things of some particular kind; pebbles, horses, inches, pounds' weight. What the name of number connotes is, the manner in which single objects of the given kind must be put together, in order to produce that particular aggregate. If the aggregate be of pebbles, and we call it *two*, the name implies that, to compose the aggregate, one pebble must be joined to one pebble. If we call it *three*, one and one and one pebble must be brought together to produce it, or else one pebble must be joined to an aggregate of the kind called *two*, already existing. The aggregate which we call *four*, has a still greater number of characteristic modes of formation. One and one and one and one pebble may be brought together; or two aggregates of the kind called *two* may be united; or one pebble may be added to an aggregate of the kind called *three*. Every succeeding number in the ascending series, may be formed by the junction of smaller numbers in a progressively greater variety of ways. Even limiting the parts to two, the number may be formed, and consequently may be divided, in as many different ways as there are numbers smaller than itself; and, if we admit of threes, fours, etc., in a still greater variety. Other modes of arriving at the same aggregate present themselves, not by the union of smaller, but by the dismemberment of larger aggregates. Thus, *three pebbles* may be formed by taking away one pebble from an aggregate of four; *two pebbles*, by an equal division of a similar aggregate; and so on.

Every arithmetical proposition; every statement of the result of an arithmetical operation; is a statement of one of the modes of formation of a given number. It affirms that a certain aggregate might have been formed by putting together certain other aggregates, or by withdrawing certain portions of some aggregate; and that, by consequence, we might reproduce those aggregates from it, by reversing the process.

Thus, when we say that the cube of 12 is 1728, what we affirm is this: that if, having a sufficient number of pebbles or of any other objects, we put them together into the particular sort of parcels or aggregates called twelves; and put together these twelves again into similar collections; and, finally, make up twelve of these largest parcels; the aggregate thus formed will be such a one as we call 1728; namely, that which (to take the most familiar of its modes of formation) may be made by joining the parcel called a thousand pebbles, the parcel called seven hundred pebbles, the parcel called twenty pebbles, and the parcel called eight pebbles.

The converse proposition that the cube root of 1728 is 12, asserts that this large aggregate may again be decomposed into the twelve twelves of twelves of pebbles which it consists of.

The modes of formation of any number are innumerable; but when we know one mode of formation of each, all the rest may be determined deductively. If we know that a is formed from b and c, b from a and e, c from d and f, and so forth, until we have included all the numbers of any scale we choose to select (taking care that for each number the mode of formation be really a distinct one, not bringing us round again to the former numbers, but introducing a new number), we have a set of propositions from which we may reason to all the other modes of formation of those numbers from one another. Having established a chain of inductive truths connecting together all the numbers of the scale, we can ascertain the formation of any one of those numbers from any other by merely traveling from one to the other along the chain. Suppose that we know only the following modes of formation: 6=4+2, 4=7-3, 7=5+2, 5=9-4. We could determine how 6 may be formed from 9. For 6=4+2=7-3+2=5+2-3+2=9-4+2-3+2. It may therefore be formed by taking away 4 and 3, and adding 2 and 2. If we know besides that 2+2=4, we obtain 6 from 9 in a simpler mode, by merely taking away 3.

It is sufficient, therefore, to select one of the various modes of formation of each number, as a means of ascertaining all the rest. And since things which are uniform, and therefore simple, are most easily received and retained by the understanding, there is an obvious advantage in selecting a mode of formation which shall be alike for all; in fixing the connotation of names of number on one uniform principle. The mode in which our existing numerical nomenclature is contrived possesses this advantage, with the additional one, that it happily conveys to the mind two of the modes of formation of every number. Each number is considered as formed by the addition of a unit to the number next below it in magnitude, and this mode of formation is conveyed by the place which it occupies in the series. And each is also considered as formed by the addition of a number of units less than ten, and a number of aggregates each equal to one of the successive powers of ten; and this mode of its formation is expressed by its spoken name, and by its numerical character.

What renders arithmetic the type of a deductive science, is the fortunate applicability to it of a law so comprehensive as "The sums of equals are equals:" or (to express the same principle in less familiar but more characteristic language), Whatever is made up of parts, is made up of the parts of those parts. This truth, obvious to the senses in all cases which can be fairly referred to their decision, and so general as to be co-extensive with nature itself, being true of all sorts of phenomena (for all admit of being numbered), must be considered an inductive truth, or law of nature, of the highest order. And every arithmetical operation is an application of this law, or of other laws capable of being deduced from it. This is our warrant for all calculations. We believe that five and two are equal to seven, on the evidence of this inductive law, combined with the definitions of those numbers. We arrive at that conclusion (as all know who remember how they first learned it) by adding a single unit at a time: 5 + 1=6, therefore 5+1+1=6+1=7; and again 2=1+1, therefore 5+2=5+1+1=7.

§ 6. Innumerable as are the true propositions which can be formed concerning particular numbers, no adequate conception could be gained, from these alone, of the extent of the truths composing the science of number. Such propositions as we have spoken of are the least general of all numerical truths. It is true that even these are co-extensive with all nature; the properties of the number four are true of all objects that are divisible into four equal parts, and all objects are either actually or ideally so divisible. But the propositions which compose the science of algebra are true, not of a particular number, but of all numbers; not of all things under the condition of being divided in a particular way, but of all things under the condition of being divided in any way--of being designated by a number at all.

Since it is impossible for different numbers to have any of their modes of formation completely in common, it is a kind of paradox to say, that all propositions which can be made concerning numbers relate to their modes of formation from other numbers, and yet that there are propositions which are true of all numbers. But this very paradox leads to the real principle of generalization concerning the properties of numbers. Two different numbers can not be formed in the same manner from the same numbers; but they may be formed in the same manner from different numbers; as nine is formed from three by multiplying it into itself, and sixteen is formed from four by the same process. Thus there arises a classification of modes of formation, or in the language commonly used by mathematicians, a classification of Functions. Any number, considered as formed from any other number, is called a function of it; and there are as many kinds of functions as there are modes of formation. The simple functions are by no means numerous, most functions being formed by the combination of several of the operations which form simple functions, or by successive repetitions of some one of those operations. The simple functions of any number x are all reducible to the following forms: $x+a$, $x-a$, ax, x/a, log. x (to the base a), and the same expressions varied by putting x for a and a for x, wherever that substitution would alter the value: to which, perhaps, ought to be added sin x, and arc (sin=x). All other functions of x are formed by putting some one or more of the simple functions in the place of x or a, and subjecting them to the same elementary operations.

In order to carry on general reasonings on the subject of Functions, we require a nomenclature enabling us to express any two numbers by names which, without specifying what particular numbers they are, shall show what function each is of the other; or, in other words, shall put in evidence their mode of formation from one another. The system of general language called algebraical notation does this. The expressions a and a2+3a denote, the one any number, the other the number formed from it in a particular manner. The expressions a, b, n, and (a+b)n, denote any three numbers, and a fourth which is formed from them in a certain mode.

The following may be stated as the general problem of the algebraical calculus: F being a certain function of a given number, to find what function F will be of any function of that number. For example, a binomial $a + b$ is a function of its two parts a and b, and the parts are, in their turn, functions of $a + b$: now ($a + b$)n is a certain function of the binomial; what function will this be of a and b, the two parts? The answer to this question is the binomial theorem. The formula *(a + b)n = an + n/1 an-1 b + n.n-1/1.2 an-2 b2*, etc., shows in what manner the number which is formed by multiplying $a + b$ into itself n times, might be formed without that process, directly from *a, b*, and n. And of this nature are all the theorems of the science of number. They assert the identity of the result of different modes of formation. They affirm that some mode of formation from x, and some mode of formation from a certain function of x, produce the same number.

Such, as above described, is the aim and end of the calculus. As for its processes, every one knows that they are simply deductive. In demonstrating an algebraical theorem, or in resolving an equation, we travel from the *datum* to the *quæsitum* by pure ratiocination; in which the only premises introduced, besides the original hypotheses, are the fundamental axioms already mentioned--that things equal to the same thing are equal to one another, and that the sums of equal things are equal. At each step in the demonstration or in the calculation, we apply one or other of these truths, or truths deducible from them, as, that the differences, products, etc., of equal numbers are equal.

It would be inconsistent with the scale of this work, and not necessary to its design, to carry the analysis of the truths and processes of algebra any further; which is also the less needful, as the task has been, to a very great extent, performed by other writers. Peacock's Algebra, and Dr. Whewell's *Doctrine of Limits*, are full of instruction on the subject. The profound treatises of a truly philosophical mathematician, Professor De Morgan, should be studied by every one who desires to comprehend the evidence of mathematical truths, and the meaning of the obscurer processes of the calculus, and the speculations of M. Comte, in his *Cours de Philosophie Positive*, on the philosophy of the higher branches of mathematics, are among the many valuable gifts for which philosophy is indebted to that eminent thinker.

§ 7. If the extreme generality, and remoteness not so much from sense as from the visual and tactual imagination, of the laws of number, renders it a somewhat difficult effort of abstraction to conceive those laws as being in reality physical truths obtained by observation; the same difficulty does not exist with regard to the laws of extension. The facts of which those laws are expressions, are of a kind peculiarly accessible to the senses, and suggesting eminently distinct images to the fancy. That geometry is a strictly physical science would doubtless have been recognized in all ages, had it not been for the illusions produced by two circumstances. One of these is the characteristic property, already noticed, of the facts of geometry, that they may be collected from our ideas or mental pictures of objects as effectually as from the objects themselves. The other is, the demonstrative character of geometrical truths; which was at one time supposed to constitute a radical distinction between them and physical truths; the latter, as resting on merely probable evidence, being deemed essentially uncertain and unprecise. The advance of knowledge has, however, made it manifest that physical science, in its better understood branches, is quite as demonstrative as geometry. The task of deducing its details from a few comparatively simple principles is found to be any thing but the impossibility it was once supposed to be; and the notion of the superior certainty of geometry is an illusion, arising from the ancient prejudice which, in that science, mistakes the ideal data from which we reason, for a peculiar class of realities, while the corresponding ideal data of any deductive physical science are recognized as what they really are, hypotheses.

Every theorem in geometry is a law of external nature, and might have been ascertained by generalizing from observation and experiment, which in this case resolve themselves into comparison and measurement. But it was found practicable, and, being practicable, was desirable, to deduce these truths by ratiocination from a small number of general laws of nature, the certainty and universality of which are obvious to the most careless observer, and which compose the first principles and ultimate premises of the science. Among these general laws must be included the same two which we have noticed as ultimate principles of the Science of Number also, and which are applicable to every description of quantity; viz., The sums of equals are equal, and Things which are equal to the same thing are equal to one another; the latter of which may be expressed in a manner more suggestive of the inexhaustible multitude of its consequences, by the following terms: Whatever is equal to any one of a number of equal magnitudes, is equal to any other of them. To these two must be added, in geometry, a third law of equality, namely, that lines, surfaces, or solid spaces, which can be so applied to one another as to coincide, are equal. Some writers have asserted that this law of nature is a mere verbal definition; that the expression "equal magnitudes" *means* nothing but magnitudes which can be so applied to one another as to coincide. But in this opinion I can not agree. The equality of two geometrical magnitudes can not differ fundamentally in its nature from the equality of two weights, two degrees of heat, or two portions of duration, to none of which would this definition of equality be suitable. None of these things can be so applied to one another as to coincide, yet we perfectly understand what we mean when we call them equal. Things are equal in magnitude, as things are equal in weight, when they are felt to be exactly similar in respect of the attribute in which we compare them: and the application of the objects to each other in the one case, like the balancing them with a pair of scales in the other, is but a mode of bringing them into a position in which our senses can recognize deficiencies of exact resemblance that would otherwise escape our notice.

Along with these three general principles or axioms, the remainder of the premises of geometry consists of the so-called definitions: that is to say, propositions asserting the real existence of the various objects therein designated, together with some one property of each. In some cases more than one property is commonly assumed, but in no case is more than one necessary. It is assumed that there are such things in nature as straight lines, and that any two of them setting out from the same point, diverge more and more without limit. This assumption (which includes and goes beyond Euclid's axiom that two straight lines can not inclose a space) is as indispensable in geometry, and as evident, resting on as simple, familiar, and universal observation, as any of the other axioms. It is also assumed that straight lines diverge from one another in different degrees; in other words, that there are such things as angles, and that they are capable of being equal or unequal. It is assumed that there is such a thing as a circle, and that all its radii are equal; such things as ellipses, and that the sums of the focal distances are equal for every point in an ellipse; such things as parallel lines, and that those lines are everywhere equally distant.(200)

§ 8. It is a matter of more than curiosity to consider, to what peculiarity of the physical truths which are the subject of geometry, it is owing that they can all be deduced from so small a number of original premises; why it is that we can set out from only one characteristic property of each kind of phenomenon, and with that and two or three general truths relating to equality, can travel from mark to mark until we obtain a vast body of derivative truths, to all appearance extremely unlike those elementary ones.

The explanation of this remarkable fact seems to lie in the following circumstances. In the first place, all questions of position and figure may be resolved into questions of magnitude. The position and figure of any object are determined by determining the position of a sufficient number of points in it; and the position of any point may be determined by the magnitude of three rectangular co-ordinates, that is, of the perpendiculars drawn from the point to three planes at right angles to one another, arbitrarily selected. By this transformation of all questions of quality into questions only of quantity, geometry is reduced to the single problem of the measurement of magnitudes, that is, the ascertainment of the equalities which exist between them. Now when we consider that by one of the general axioms, any equality, when ascertained, is proof of as many other equalities as there are other things equal to either of the two equals; and that by another of those axioms, any ascertained equality is proof of the equality of as many pairs of magnitudes as can be formed by the numerous operations which resolve themselves into the addition of the equals to themselves or to other equals; we cease to wonder that in proportion as a science is conversant about equality, it should afford a more copious supply of marks of marks; and that the sciences of number and extension, which are conversant with little else than equality, should be the most deductive of all the sciences.

There are also two or three of the principal laws of space or extension which are unusually fitted for rendering one position or magnitude a mark of another, and thereby contributing to render the science largely deductive. First, the magnitudes of inclosed spaces, whether superficial or solid, are completely determined by the magnitudes of the lines and angles which bound them. Secondly, the length of any line, whether straight or curve, is measured (certain other things being given) by the angle which it subtends, and *vicè versa*. Lastly, the angle which any two straight lines make with each other at an inaccessible point, is measured by the angles they severally make with any third line we choose to select. By means of these general laws, the measurement of all lines, angles, and spaces whatsoever might be accomplished by measuring a single straight line and a sufficient number of angles; which is the plan actually pursued in the trigonometrical survey of a country; and fortunate it is that this is practicable, the exact measurement of long straight lines being always difficult, and often impossible, but that of angles very easy. Three such generalizations as the foregoing afford such facilities for the indirect measurement of magnitudes (by supplying us with known lines or angles which are marks of the magnitude of unknown ones, and thereby of the spaces which they inclose), that it is easily intelligible how from a few data we can go on to ascertain the magnitude of an indefinite multitude of lines, angles, and spaces, which we could not easily, or could not at all, measure by any more direct process.

§ 9. Such are the remarks which it seems necessary to make in this place, respecting the laws of nature which are the peculiar subject of the sciences of number and extension. The immense part which those laws take in giving a deductive character to the other departments of physical science, is well known; and is not surprising, when we consider that all causes operate according to mathematical laws. The effect is always dependent on, or is a function of, the quantity of the agent; and generally of its position also. We can not, therefore, reason respecting causation, without introducing considerations of quantity and extension at every step; and if the nature of the phenomena admits of our obtaining numerical data of sufficient accuracy, the laws of quantity become the grand instrument for calculating forward to an effect, or backward to a cause. That in all other sciences, as well as in geometry, questions of quality are scarcely ever independent of questions of quantity, may be seen from the most familiar phenomena. Even when several colors are mixed on a painter's palette, the comparative quantity of each entirely determines the color of the mixture.

With this mere suggestion of the general causes which render mathematical principles and processes so predominant in those deductive sciences which afford precise numerical data, I must, on the present occasion, content myself; referring the reader who desires a more thorough acquaintance with the subject, to the first two volumes of M. Comte's systematic work.

In the same work, and more particularly in the third volume, are also fully discussed the limits of the applicability of mathematical principles to the improvement of other sciences. Such principles are manifestly inapplicable, where the causes on which any class of phenomena depend are so imperfectly accessible to our observation, that we can not ascertain, by a proper induction, their numerical laws; or where the causes are so numerous, and intermixed in so complex a manner with one another, that even supposing their laws known, the computation of the aggregate effect transcends the powers of the calculus as it is, or is likely to be; or, lastly, where the causes themselves are in a state of perpetual fluctuation; as in physiology, and still more, if possible, in the social science. The mathematical solutions of physical questions become progressively more difficult and imperfect, in proportion as the questions divest themselves of their abstract and hypothetical character, and approach nearer to the degree of complication actually existing in nature; insomuch that beyond the limits of astronomical phenomena, and of those most nearly analogous to them, mathematical accuracy is generally obtained "at the expense of the reality of the inquiry:" while even in astronomical questions, "notwithstanding the admirable simplicity of their mathematical elements, our feeble intelligence becomes incapable of following out effectually the logical combinations of the laws on which the phenomena are dependent, as soon as we attempt to take into simultaneous consideration more than two or three essential influences."(201) Of this, the problem of the Three Bodies has already been cited, more than once, as a remarkable instance; the complete solution of so comparatively simple a question having vainly tried the skill of the most profound mathematicians. We may conceive, then, how chimerical would be the hope that mathematical principles could be advantageously applied to phenomena dependent on the mutual action of the innumerable minute particles of bodies, as those of chemistry, and still more, of physiology; and for similar reasons those principles remain inapplicable to the still more complex inquiries, the subjects of which are phenomena of society and government.

The value of mathematical instruction as a preparation for those more difficult investigations, consists in the applicability not of its doctrines, but of its method. Mathematics will ever remain the most perfect type of the Deductive Method in general; and the applications of mathematics to the deductive branches of physics, furnish the only school in which philosophers can effectually learn the most difficult and important portion of their art, the employment of the laws of simpler phenomena for explaining and predicting those of the more complex. These grounds are quite sufficient for deeming mathematical training an indispensable basis of real scientific education, and regarding (according to the *dictum* which an old but unauthentic tradition ascribes to Plato) one who is {~GREEK SMALL LETTER ALPHA WITH PSILI~}{~GREEK SMALL LETTER GAMMA~}{~GREEK SMALL LETTER EPSILON~}{~GREEK SMALL LETTER OMEGA~}{~GREEK SMALL LETTER MU~}{~GREEK SMALL LETTER EPSILON WITH OXIA~}{~GREEK SMALL LETTER TAU~}{~GREEK SMALL LETTER RHO~}{~GREEK SMALL LETTER ETA~}{~GREEK SMALL LETTER TAU~}{~GREEK SMALL LETTER OMICRON~}{~GREEK SMALL LETTER FINAL

SIGMA~}, as wanting in one of the most essential qualifications for the successful cultivation of the higher branches of philosophy.